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LETTER TO THE EDITOR

Exact results for 2D directed animals on a strip of finite width

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Abstract. We prove a conjecture giving the exact number of directed animals of s sites with any root, on a strip of finite width of a square lattice. We also rederive more simply some previous results concerning the connective constant and particular eigenvectors of the transfer matrix.

The problem of directed lattice animals has recently attracted much theoretical attention (Day and Lubensky 1982, Dhar *et al* 1982, Redner and Yang 1982). In particular, Nadal *et al* (1982) have found some exact results for two-dimensional animals living on a strip of finite width. Namely, they derived the expression of the connective constant and of the eigenvector of the transfer matrix when the eigenvalue is one (see (i) and (ii) below). They also proposed a conjecture ((iii) below) for the number of animals of s sites on a strip of width n which is a generalisation of a previous expression guessed by Dhar *et al* (1982). We give in this letter a simple proof of this general conjecture. In the special case of animals of infinite width beginning with only one occupied site, the result has been obtained by Dhar (1982) by relating 2D animals and a hard square model previously studied (Verhagen 1976, Baxter 1981). We also rederive more straightforwardly the other two exact results of Nadal *et al.*

We begin by briefly reviewing the formalism and notation used in Nadal *et al.* We consider directed site animals on a strip of finite width n of a square lattice. The preferred direction lies along a diagonal of the lattice. The strip is infinite along this preferred direction and has periodic boundary conditions in the perpendicular direction (see figure 1). We call $\Omega_s(C)$ the number of directed animals consisting of s sites and beginning with a root C. C represents a given set of occupied sites at column 0 and any animal which is counted in $\Omega_s(C)$ is a cluster of s sites with the following property: any site of the cluster can be reached from at least one site of the root by a path which never goes opposite to the preferred direction. For s sufficiently large $\Omega_s(C)$ is expected to have a simple asymptotic form:

$$\Omega_s(C) \sim \mu_0^s a(C). \quad (1)$$

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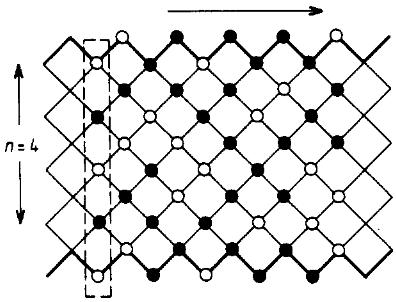

Figure 1. A directed animal on a strip of a square lattice. The periodic boundary conditions are realised by the identification of the two heavy lines. The horizontal arrow indicates the preferred direction. The root of the animal is the configuration on the first row, which is in the broken rectangle. ● Occupied site; ○ empty site.

The purpose of this letter is to give simple proofs of the three following facts:

(i) $\mu_0 = 1 + 2\cos(\pi/2n)$;

(ii) $a(C) = \prod_{i=1}^{n-1} \left(\frac{\sin[(i+\frac{1}{2})\pi/2n]}{\sin(\pi/4n)}\right)^{N_i}$;

(iii) $\Omega_s(C) = \frac{1}{n}\sum_p (-1)^p \sin\alpha_p \prod_{i=1}^{n-1} \left(\frac{\sin[(i+\frac{1}{2})\alpha_p]}{\sin(\frac{1}{2}\alpha_p)}\right)^{N_i} (1+2\cos\alpha_p)^{s-1}$,

where $\alpha_p = (2p+1)\pi/2n$ and the N_i are the number of holes of i consecutive sites in the root C.

Let us define $\tilde{M}$ by

$\tilde{M}(C, C') = 1$ if the configuration C' is allowed to follow configuration C,

$\tilde{M}(C, C') = 0$ otherwise.

If we call M the restriction of $\tilde{M}$ to the space of non-empty roots ($C \neq C_0$), $\Omega_s(C)$ satisfies the following recursion relation:

$$\Omega_{s+m(C)}(C) = \sum_{C'} M(C, C')\Omega_s(C'), \qquad s \geq 1, \tag{2}$$

where $m(C)$ is the number of occupied sites of the root C. If we impose the additional conditions

$$\Omega_s = 0, \qquad 1 \leq s < m(C), \tag{3a}$$

$$\Omega_{m(C)}(C) = 1, \tag{3b}$$

the recursion relation (2) allows us to calculate $\Omega_s(C)$ without upper limit for s. Therefore, (2) and ($3a, b$) define $\Omega_s(C)$ uniquely. The transfer matrices $\tilde{T}_\mu$ and T_μ are defined by

$$\tilde{T}_\mu(C, C') = (1/\mu)^{m(C)}\tilde{M}(C, C'), \qquad T_\mu = (1/\mu)^{m(C)}M(C, C'). \tag{4}$$

The connective constant μ_0 is the largest value of μ for which the largest eigenvalue of T_μ is one (Nadal *et al* 1982). The components of the eigenvector corresponding to this eigenvalue are $a(C)$ in the root base, as is readily seen by putting (1) into (2).

We represent a root by a sequence of n $\frac{1}{2}$-spins, occupied sites being up spins and empty sites being down spins. Then it is not difficult to see that one can rewrite T as an operator acting in this space of spins:

$$\tilde{T}_\mu = \mathrm{Tr}_a \overset{\curvearrowleft}{\prod_{j=1,n}} T_j \qquad \text{where } T_j = \begin{pmatrix} A_j & B_j \\ \mu^{-1}D_j & \mu^{-1}D_j \end{pmatrix}. \tag{5}$$

The arrow means that the T_j are written from right to left with increasing j and the subscript a means that the trace and ordered product are to be taken in the auxiliary space (where T_j is a 2×2 matrix) and *not* in the spin internal spaces where the matrices A, B, D are operators

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}. \tag{6}$$

Now $a(C, \alpha)$ defined by

$$\begin{aligned} a(C,\alpha) &= \prod_{i=1}^{n-1} \left(\frac{\sinh[(i+\frac{1}{2})\alpha]}{\sinh \frac{1}{2}\alpha} \right)^{N_i}, \qquad C \neq C_0, \\ a(C_0,\alpha) &= 2\cosh n\alpha, \end{aligned} \tag{7}$$

may be rewritten as

$$a(C,\alpha) = \mathrm{Tr} \overset{\curvearrowleft}{\prod_{K=1}^{n}} M_{\sigma_K}(\alpha, \tfrac{1}{2}(1+\coth \tfrac{1}{2}\alpha)) \tag{8}$$

where $\sigma_K = \pm 1$ is the spin at site K of the root C and M_+, M_- are two matrices:

$$M_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad M_-(\alpha, u) = \begin{pmatrix} u\,\mathrm{e}^{\alpha} + (1-u)\,\mathrm{e}^{-\alpha} & (1-u)(\mathrm{e}^{-\alpha} - \mathrm{e}^{\alpha}) \\ u(\mathrm{e}^{-\alpha} - \mathrm{e}^{\alpha}) & u\,\mathrm{e}^{-\alpha} + (1-u)\,\mathrm{e}^{\alpha} \end{pmatrix}.$$

In order to check (8), it is sufficient to use the three properties

$$\begin{aligned} &M_+^p = M_+, \\ &M_-^p(\alpha, u) = M_-(p\alpha, u), \\ &\mathrm{Tr}(M_+^{p_1} M_-^{q_1} \ldots M_+^{p_k} M_-^{q_k}) = \mathrm{Tr}(M_+ M_-^{q_1}) \times \ldots \times \mathrm{Tr}(M_+ M_-^{q_k}). \end{aligned}$$

We now search for the eigenvector $|\psi(\alpha)\rangle$ of T, in the form

$$|\psi_{(\alpha)}\rangle \equiv \sum_C a(C,\alpha)|C\rangle = \mathrm{Tr}_a\left(\overset{\curvearrowleft}{\prod_{j=1}^{n}} Z_j \right)|C_0\rangle$$

with $Z_j = M_+ D_j + M_- B_j$.

Taking inspiration from a standard technique of integrable systems (Baxter 1982, Faddeev 1980), we try to find a 4×4 matrix R acting in $a \otimes a$ such that

$$R^{-1}\left(\tilde{T}_j \underset{a}{\otimes} Z_j \right) R = \begin{pmatrix} Z_j & 0 \\ X & 0 \end{pmatrix} \tag{9}$$

where X may be any operator. Indeed, the existence of R ensures that $|\psi(\alpha)\rangle$ is an eigenvector of $\tilde{T}$ with eigenvalue one:

$$\tilde{T}_\mu|\Psi_\alpha\rangle = \mathrm{Tr}_a\left(\overset{\curvearrowleft n}{\prod_{j=1}} \tilde{T}_j\right) \mathrm{Tr}_a\left(\overset{\curvearrowleft n}{\prod_{j=1}} Z_j\right)|C_0\rangle = \mathrm{Tr}_{a\otimes a}\left(\overset{\curvearrowleft n}{\prod_{j=1}} \tilde{T}_j \underset{a}{\otimes} Z_j\right)|C_0\rangle$$

$$= \mathrm{Tr}_{a\otimes a}\left(\overset{\curvearrowleft n}{\prod_{j=1}} R^{-1}\tilde{T}_j \underset{a}{\otimes} Z_j R\right)|C_0\rangle = \mathrm{Tr}_a\left(\overset{\curvearrowleft n}{\prod_{j=1}} Z_j\right)|C_0\rangle = |\psi_\alpha\rangle.$$

By taking the trace in a on both sides of (9), R disappears and one sees that the equality cannot be true unless

$$\mu = 1 + 2\cosh\alpha. \tag{10}$$

In this case, R can be found. For example, a good choice is

$$R = \begin{pmatrix} M_-(-\alpha/2, u) & -1 - M_-(-\alpha, u)M_+ \\ 0 & 1 \end{pmatrix}.$$

Thus we have proved that $|\psi(\alpha)\rangle$ is an eigenvector of eigenvalue one of $\tilde{T}_\mu$ if μ and α are related by (10). It is also an eigenvector of T_μ with eigenvalue one only if its component $\cosh(n\alpha)$ on $|C_0\rangle$ vanishes. Thus every $|\psi(\mathrm{i}\alpha_p)\rangle$, where $\alpha_p = (2p+1)\pi/2n$, is an eigenvector of T_{μ_p} with eigenvalue one. However, it is only if $\alpha = \alpha_0$ and $\mu = 1 + 2\cos\alpha_0$ that all the $a(C, \alpha)$ are positive and that the largest eigenvalue of the transfer matrix is equal to 1. This concludes the demonstration of (i) and (ii).

We proceed to prove the conjecture (iii). We observe that $\mu^s_p a(C, \alpha_p)$ is a solution of the recurrence relations (2) because $|\psi(\mathrm{i}\alpha_p)\rangle$ is an eigenvector of T_{μ_p} with eigenvalue one. If we find a linear combination of these solutions $\Sigma_{p=0}^{n-1} \lambda_p \mu^s_p a(C, \alpha_p)$ which verifies the initial conditions ($3a, b$) we can conclude that

$$\Omega_s(C) = \sum_{p=0}^{n-1} \lambda_p \mu^s_p a(C, \mathrm{i}\alpha_p).$$

We are going now to show that

$$\lambda_p = (-1)^p (\sin\alpha_p)/(1 + 2\cos\alpha_p)$$

satisfies all the requirements. This will complete the proof of (iii). Indeed, consider the meromorphic function $f(Z)$,

$$f(Z) = \frac{Z^{n-1}}{Z^{2n}+1}\,\frac{1}{Z^2+Z+1}\left\{\mathrm{Tr}\overset{\curvearrowleft n}{\prod_{i=1}}\left[M_-(Z) + X_i\left[1 + Y_i\left(Z + \frac{1}{Z} + 1\right)\right]M_+\right]\right.$$

$$\left. - \mathrm{Tr}\overset{\curvearrowleft n}{\prod_{i=1}}[M_-(Z) + X_i M_+]\right\},$$

where $M_-(Z)$ is a straightforward continuation in the complex plane of $M_-(\alpha, \frac{1}{2}(1 + \coth\frac{1}{2}\alpha))$

$$M_-(Z) = \begin{pmatrix} 1 + Z + 1/Z & 1 + 1/Z \\ -(1+Z) & -1 \end{pmatrix} = \frac{1}{Z} M_A(Z).$$

The fact that X_i, Y_i are arbitrary will allow us to prove all the conditions ($3a, b$) at once. We readily see the analytic structure of f:

(a) f decreases at infinity like Z^{-3};

(b) f has poles of order one at:

0, with a residue R,

$$R = \mathrm{Tr} \prod_{i=1}^{\overset{\curvearrowleft}{n}} (M_A(0) + X_i Y_i M_+) - \mathrm{Tr} \prod_{i=1}^{\overset{\curvearrowleft}{n}} M_A(0) = \prod_{i=1}^{\overset{\curvearrowleft}{n}} (1 + X_i Y_i) - 1;$$

$Z_p = \exp(\mathrm{i}\alpha_p)$, $\alpha_p = \dfrac{2p+1}{2n}\pi$, $0 \leqslant p \leqslant 2n-1$, with a residue

$$R_p = -\frac{1}{2n}\mathrm{i}(-1)^p \frac{\cos\alpha_p - \mathrm{i}\sin\alpha_p}{1 + 2\cos\alpha_p}$$
$$\times \left(\mathrm{Tr} \prod_{i=1}^{\overset{\curvearrowleft}{n}} (M_-(Z_p) + X_i(1 + Y_i\mu_p)M_+) - \mathrm{Tr} \prod_{i=1}^{\overset{\curvearrowleft}{n}} (M_-(Z_p) + X_i M_+) \right).$$

It may seem to the reader that there are two other poles when $Z = \exp \pm (\mathrm{i}2\pi/3)$, but he should notice that the expression in parentheses vanishes at these values. (a) implies that the sum of the residues vanishes:

$$-\sum_{p=0}^{2n-1} R_p = \sum_{p=0}^{n-1} \frac{\sin\alpha_p}{1 + 2\cos\alpha_p}(-1)^p$$
$$\times \left(\mathrm{Tr} \prod_{i=1}^{\overset{\curvearrowleft}{n}} (M_-(\alpha_p) + X_i(1 + Y_i\mu_p)M_+) - \mathrm{Tr} \prod_{i=1}^{\overset{\curvearrowleft}{n}} (M_-(\alpha_p) + X_i M_+) \right)$$
$$= \prod_{i=1}^{n} (1 + X_i Y_i) - 1 \tag{11}$$

(we have added together R_p and R_{2n-p-1}). Equation (11) is an equality between polynomials in $2n$ unknowns. The equality of the polynomials implies the equality of their coefficients. If we take the coefficient of $\Pi_{i\in C} X_i$ and remember (8) we find

$$\frac{1}{n}\sum_{p=0}^{n-1} (-1)^p \frac{\sin\alpha_p}{1 + 2\cos\alpha_p}\left(\prod_{i\in C} (1 + \mu_p Y_i) - 1 \right) a(C, \mathrm{i}\alpha_p) = 1.$$

This implies now

$$\frac{1}{n}\sum_{p=0}^{n-1} (-1)^p \sin\alpha_p (1 + 2\cos\alpha_p)^{s-1} a(C, \mathrm{i}\alpha_p) = \begin{cases} 1 & \text{if } s = m(C), \\ 0 & \text{if } 1 \leqslant s < m(C), \end{cases}$$

and concludes the proof of (iii).

As is well known, $\nu_\perp = \theta = \frac{1}{2}$ (Breuer and Janssen 1982, Cardy 1982) is a consequence of (iii) (Nadal *et al* 1982, Dhar 1982). The last exponent of interest, $\nu_\parallel \simeq \frac{9}{11}$ which gives the average asymptotic animal length, has not been computed analytically so far. This would require a generalisation of the method in order to find other eigenvectors of T, or at least its left eigenvector for the eigenvalue one, since the matrix T is not symmetric.

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Note added. The choice of periodic boundary conditions for the strip is not essential. It is, for example, possible to use the same techniques for a strip with free boundary conditions (i.e. the strip of figure 1 is cut along the two heavy lines with the site 1 at the bottom of the root and the site $n = 5$ at the top). The boundary condition is taken into account by adding a matrix under every trace. One can then prove in complete analogy with the main result of this letter that

$$\Omega_s(C) = \frac{1}{n+1} \sum_{p=0}^{n-1} \left[(-1)^p \sin \alpha_p (1 + 2\cos \alpha_p)^{S-1} \times \left(\frac{\cos(h_n + \frac{1}{2})\alpha_p - \cos(h_n + \frac{3}{2})\alpha_p}{\sin \frac{1}{2}\alpha_p} \right) \prod_{i=1}^{n-1} \left(\frac{\sin(i + \frac{1}{2})\alpha_p}{\sin \frac{1}{2}\alpha_p} \right)^{N_i} \right]$$

where now $\alpha_p = (p+1)\pi/(n+1)$ and h_n is the number of sites in the hole which *ends* at the boundary (if it exists). This hole must not be counted in the N_i's.

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